MODIFIED LATIN SQUARE TYPE PBIB DESIGNS

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1. INTRODUCTION

In this paper, we give a three associate-class association scheme, by slightly modifying the definition of $L_i(s)$ association scheme given by Bose and Shimamoto (1952) for the s^2 symbols. We shall call the new association scheme as modified Latin-square-type ($ML_i(s)$) association scheme with i constraints and the corresponding PBIB designs as $ML_i(s)$ designs. Two series of $ML_i(s)$ designs with their usefulness as confounded s^2 symmetrical factorial experiments is also discussed, therein. For the definitions of the various statistical terms henceforth used, we refer to Raghavarao (1971).

2. $ML_i(s)$ DESIGNS

We define $ML_i(s)$ association scheme for the s^2 symbols as following :

Definition 2.1. Let s^2 symbols be arranged in $s \times s$ square array

$$
\begin{array}{ccccc}
1 & 2 & \ldots & \ldots & s \\
s+1 & s+2 & \ldots & \ldots & 2s \\
- & - & \ldots & \ldots & - \\
 & & & & \\
- & - & \ldots & \ldots & - \\
- & - & \ldots & \ldots & - \\
- & - & \ldots & \ldots & - \\
s^2-s+1 & s^2-s+2 & \ldots & \ldots & s^2
\end{array} \qquad (2.1)
$$

Let $(i-2)$ mutually orthogonal latin-squares (MOLS) of orders exist, Let these $(i-2)$ MOLS be superimposed on this square array. Two symbols will be called

(1) first associates, if they occur in the same row or column of the array ;

(2) second associates, if they occur in positions occupied by the same letter in any of the (i − 2) MOLS; and

(3) third associates, otherwise.

The parameters of the $ML_i(s)$ association scheme will be

$n_i=2(s-1),\ n_2=(i-1)(s-2),\ n_3=(s-1)(s-i+1),$

$$P_1=\begin{bmatrix} (s-2) & (i-2) & (s-i+1) \\ (i-2) & (i-2)(i-3) & (i-2)(s-i+1) \\ (s-i+1) & (i-2)(s-i+1) & (s-i)(s-i+1) \end{bmatrix},$$

$$(2.2)\quad P_2=\begin{bmatrix} 2 & 2(i-3) & 2(s-i+1) \\ 2(i-3) & (s-2)+(i-3)(i-4) & (i-3)(s-i+1) \\ 2(s-i+1) & (i-3)(s-i+1) & (s-i)(s-i+1) \end{bmatrix},$$

$$P_3=\begin{bmatrix} 2 & 2(i-2) & 2(s-i) \\ 2(i-2) & (i-2)(i-3) & (i-2)(s-i) \\ 2(s-i) & (s-i)(i-2) & (s-i)^2+(i-2) \end{bmatrix}.$$

Let N be the incidence matrix of a $ML_i(s)$ design with the parameters $v, b, r, k, \lambda_1, \lambda_2, \lambda_3$. It can be verified that the characteristic roots of NN' will be $\theta_0=rk$, $\theta_1=r+(s-2)\lambda_1-(i-2)\lambda_2-(s-i+1)\lambda_3$, $\theta_2=r-2\lambda_1+(s-i+2)\lambda_2-(s-i+1)\lambda_3$, $\theta_3=r-2\lambda_1-(i-2)\lambda_2+(i-1)\lambda_3$, with their respective multiplicities $a_0=1$, $a_1=2(s-1)$, $a_2=(i-2)(s-1)$, $a_3=(s-1)(s-i+1)$.

3. Construction Methods of $ML_i(s)$ Designs

We prove the following theorem :

Theorem 3.1 : A series of $ML_i(s)$ designs with the parameters (3.1) $v=s^2$, $b=(i-2)_s$, $r=(i-2)$, $k=s$, $\lambda_1=0, \lambda_2=1$, $\lambda_3=0$, can be constructed when (i − 2) MOLS exist.

Proof : Let each of the (i−2) MOLS be superimposed on the $s\times s$ array given in (2.1) of the s^2 symbols of the $ML_i(s)$ association scheme. Then the (i−2)s sets, each of s symbols, formed by symbols corresponding to the different letters of the latin squares, constitute the series of $ML_i(s)$ designs with the parameters given in (3.1).

We define pseudo $ML_s(s)$ association scheme as following :

Definition 3.1. Let a latin square be superimposed on the $s\times s$ array given in (2.1) of s^2 symbols. Two symbols will be called

(1) first associates, if they occur in the same row or column of the array ;

(2) third associates, if they occur in positions occupied by the same letter of the latin square ; and

(3) second associates, otherwise.

The parameters of the pseudo ML_s(s) association scheme, will be the same as given in (2.2), by taking i=s. The pseudo ML_s(s) association scheme will be the usual ML_s(s) association scheme when s is a prime or a prime power. The PBIB designs with the pseudo ML_s(s) association scheme will be called the pseudo ML_s(s) designs.

We prove another theorem as following :

Theorem 3.2 : A pseudo ML_s(s) design with the parameters (3.2) $v=b=s^2$, $r=s-1=k$, $\lambda_1=0$, $\lambda_2=1$, $\lambda_3=0$ can be constructed when (s+1) is a prime or a prime power.

Proof : Let s^2 symbols be arranged in an $s\times s$ array. As (s+1) is a prime or prime power, s MOLS of order (s+1) will exist. Out of these s MOLS, (s−1) latin squares can always be written in the form having different letters in the diagonal positions and one latin square will have all zeroes in the diagonal positions.* Let the first s rows and the first s columns of these MOLS be superimposed on the $s\times s$ array. Then s^2+s sets, s^2 sets being of the size (s−1) and s sets being of the size s, can be formed by the symbols corresponding to different letters of the latin squares. Deleting s sets each of size s from these s^2+s sets, we get the pseudo ML_s(s) design with the parameters given in (3.2).

Illustration : Let us construct $ML_4(4)$ design with the parameters

(3.3) $$v=16=b,\ r=3=k,\ \lambda_1=0,\ \lambda_2=1,\ \lambda_3=0.$$

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*Let L_i (i=1, 2,..., s) be s MOLS of order (s+1). Let x be a primitive root of GF (s+1). Then as reported in Raghavarao (1971)

$$L_i=\begin{array}{ccccc} 0 & x^{i-1} & x^{s-1} & 1 & x^{i-2} \\ 1 & 1+x^{i-1} & 1+x^{s-1} & 2 & 1+x^{i-2} \\ x & x+x^{i-1} & x+x^{s-1} & x+1 & x+x^{i-2} \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ x^{s-1} & x^{s-1}+x^{i-1} & 2x^{s-1} & x^{s-1}+1 & x^{s-1}+x^{i-2} \end{array},\quad i=1, 2,..., s.$$

Then the diagonal elements of the latin-square L_i will be $(1+x^{i-1})$ $(0, 1, x, \ldots, x^{s-1})$, i=1, 2, . . ., s. Clearly all the diagonal elements will be distinct except in the case when $1+x^{i-1}=0$.

Let the 16 symbols be represented by

(3.4)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

The four MOLS of order 5 in the form mentioned in the proof of the theorem 3.2 are

(3.5)

0	1	2	4	3	0	2	4	3	1
1	2	3	0	4	1	3	0	4	2
2	3	4	1	0 ,	2	4	1	0	3 ,
4	0	1	3	2	4	1	3	2	0
3	4	0	2	1	3	0	2	1	4

0	4	3	1	2	0	3	1	2	4
1	0	4	2	3	1	4	2	3	0
2	1	0	3	4 ,	2	0	3	4	1 .
4	3	2	0	1	4	2	0	1	3
3	2	1	4	0	3	1	4	0	2

Superimposing the first four rows and first four columns of these MOLS on the 4×4 square array given in (3.4), we get the following 16 sets each of size 3 symbols, after deleting 4 sets each of size 4 :

(3.6)

(1, 14, 8), (9, 6, 3), (13, 11, 4), (10, 7, 16), (1, 7, 12), (5, 14, 11), (9, 2, 16), (6, 15, 4),

(5, 10, 4), (9, 15, 8), (13, 2, 7), (14, 3, 12), (1, 10, 15), (5, 3, 16), (13, 6, 12), (2, 11, 8).

These 16 sets constitute the $ML_4(4)$ design with the parameters given in (3.3).

4. Confounded s^2 Symmetrical Factorial Experiments

Let s^2 treatment combinations of the s^2 symmetrical factorial experiments in factors A, B, each at 0, 1,..., $(s-1)$ levels, represent the s^2 symbols of the $ML_i(s)$ association scheme. Let $s \times s$ square array be

$$
(4.1) \qquad
\begin{array}{cccc}
00 & 01 & 02 & 0\,(s-1) \\
10 & 11 & 12 & 1\,(s-1) \\
- & - & - & - \\
 & & & \\
- & - & - & - \\
- & - & - & - \\
(s-1)\,0 & (s-1)\,1 & (s-1)\,2 & (s-1)\,(s-1)
\end{array}
$$

A latin square of order s when superimposed on this array will partition (by forming sets of treatment combinations corresponding to the same letter of the latin square) the s^2 treatment combinations into s sets each of s treatment cominations confounding $(s-1)$ degrees of freedom pertaining to the interaction AB. We shall say that the confounded $(s-1)$ degrees of freedom belong to that particular latin square. Thus when $(i-2)$ MOLS are available, we can have $(i-2)$ replications of the s^2 symmetrical factorial experiment confounding $(i-2)\,(s-1)$ degrees of freedom pertaining to the interaction AB or belonging to $(i-2)$ MOLS (comparisons for each of these d.f. will be mutually othogonal) and the oth er $(s-1)\,(s-i+1)$ degrees of freedom pertaining to the interaction AB will remain unconfounded. The series of $ML_i(s)$ designs given in Theorem 3.1 can be used as the confounded s^2 symmetrical factorial experiments.

Following Shah (1958), the relative loss of information on each of $(i-2)\,(s-1)$ degrees of freedom belonging to the interacion AB or belonging to $(i-2)$ MOLS, will be $1/(i-2)$ and the remaining $(s-1)$ $(s-i+1)$ degrees of freedom belonging to the interaction AB and $2(s-1)$ degrees of freedom belonging to the main effects A, B remain unconfounded.

An advantage of the new series of $ML_i(s)$ designs given in Theorem 3.1 is that the s^2 symmetrical factorial experiments can be constructed in smaller number of replications.

The series of pseudo $ML_s(s)$ designs can also be used as the confounded s^2 experiments in $(s-1)$ replications when $(s+1)$ is a prime or a prime power. The relative loss of information on each of $2(s-1)$ degrees of freedom of the main effects A, B is $1/(s-1)^2$ and on each of $(s-2)\,(s-1)$ degrees of freedom other than $(s-1)$ degrees of freedom pertaining to the interaction AB is $(s+1)/(s-1)^2$ and on each of these $(s-1)$ degrees of freedom is $1/(s-1)^2$. These $(s-1)$ degrees of freedom pertaining to the interaction AB, will be carried by the s sets each of s treatment combinations deleted to obtain the s^2 sets in the proof of Theorem 3.2. The pseudo $ML_6(6)$ and pseudo $ML_{10}(10)$ designs with the parameters

(4.2) $$v = 36 = b, \quad r = 5 = k, \ \lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = 0$$

and

(4.3) $$v = 100 = b, \quad r = 9 = k, \ \lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = 0$$

are useful PBIB designs which can be taken as the confounded 6^2 and 10^2 symmetrical factorial experiments.

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